Gauß's Lattice Point Problem(s) Revisited
An invitation

Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

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1 Introduction to the original Problem

In analytic number theory there is an abundance of unsolved problems. One of them is Gauß's Lattice Point Problem for the circle (see Gauß [6, pp. 269–291 (in particular p. 280)] or [7, p. 657]). Hejhal once wrote that this problem might very well be more difficult than the Riemann Hypothesis (cf. [10]). Let's recall what this problem is all about: For real $t \geq 0$ let $P(t)$ denote the number of lattice Points (x, y) in the circular disc $x^2 + y^2 \leq t$ (note that the radius of this disc is $\sqrt{t}$), and let $A(t)$ be the Area ($= \pi t$) of the disc. The problem is to estimate the 'Error' $E(t) := P(t) - A(t)$ as $t \to \infty$. In [20, 19, 5] we find some history of this subject. The ultimate goal is to determine the infimum (θ) of all α satisfying $E(t) = \mathcal{O}(t^\alpha)$. Till today this is still an unsolved problem. It is clear that all (local) extremes of $E(t)$ occur at the points $t = n(\pm 0)$, so that we may restrict ourselves to the determination of $P(n)$ with $n \in \mathbf{N}$. In the past various (numerical) attempts have been made to get an impression of what θ might be. See [3, 12, 15, 17]. Various methods have been applied: Gauß's original root method[6], Tromp's step method[17]. So far Tromp's method has by far been superior in speed.

Writing $|E(t)| \leq C_\epsilon t^{\theta+\epsilon}$, the best bounds on θ are $\frac{1}{4} \leq \theta \leq \frac{131}{416} \approx 0.314904$ (cf. [11]). It was Van der Corput[2] who was the first to prove that $\theta < \frac{1}{3}$.

Experimental results suggest that $|E(t)| = o(t^{\frac{1}{4}} \log t)$ (as conjectured in [15] and confirmed in [17]).

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We propose to introduce a new method for the computation of $P(n)$, to wit: a fast *sieve* method based on [8, pp. 241–243, Section 16.9, formula (16.9.5), Theorem 278], which we already announced in [14, Section 7].

We have already written the main features of a program in Delphi Object Pascal and a Fortran version is in progress.

2 Generalization

Now we change notation: $P_2(t)$ will now denote the $P(t)$ of Section 1. Similarly $A_2(t) = A(t)$, $E_2(t) = E(t)$ and $\theta_2 = \theta$. The index 2 refers to the dimension (of the plane). We now define (in 3 dimensional space) $P_3(t)$ as the number of lattice points (x, y, z) satisfying $x^2 + y^2 + z^2 \leq t$, $V_3(t) :=$ the Volume of the pertinent sphere $= \frac{4}{3}\pi t^{\frac{3}{2}}$, and $E_3(t) := P_3(t) - V_3(t)$. Of course, also here the problem is to estimate the size of $E_3(t)$ as $t \to \infty$. Writing $|E_3(t)| \leq C_\epsilon t^{\theta_3+\epsilon}$, at the moment of writing the best bound on θ_3 is $\theta_3 \leq \frac{17}{28} \approx 0.607143$ (cf. [1]). Previous theoretical results can be found in [9, 18]. For earlier numerical work see [3, 16].

Since $P_2(t)$ can now be computed very fast, it seems worthwhile to have a go at $E_3(t)$. Summation over horizontal slices of the sphere yields $P_3(n) = P_2(n) + 2\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} P_2(n-k^2)$. However, here is a nasty catch: the values of $P_2(n)$ must be saved, which is rather demanding on fast memory. A simple back of the envelope calculation suggests that at least memory-wise $n \leq 10^{10}$ is feasible using readily available 128–256 GB machines.

3 Invitation

We would like to *invite* the golden-ager (hopefully with a lot of time) to join the crowd in an attempt to extend the computations on $E_3(t)$.

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